Angle Optimization of Graphs Embedded in the Plane

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Abstract: In this paper we study problems of drawing graphs in the plane using edge length constraints and angle optimization. Specifically we consider the problem of maximizing the minimum angle, the *MMA problem*. We solve the *MMA* problem using a spring-embedding approach where two forces are applied to the vertices of the graph: a force optimizing edge lengths and a force optimizing angles. We solve analytically the problem of computing an optimal displacement of a graph vertex optimizing the angles between edges incident to it if the degree of the vertex is at most three. We also apply a numerical approach for computing the forces applied to vertices of higher degree. We implemented our algorithm in Java and present drawings of some graphs.

1 Introduction

Angular resolution is one of the aesthetic criteria measuring the quality of graph drawings in terms of human comprehension. The *angular resolution* of a straight-line drawing in the plane is the minimum angle between any two incident edges. The study of graph drawing with angular resolution started by Formann *et al.*[10] in 1990. They introduced the *angular resolution* of a graph as the supremum angular resolution over all straight-line drawings of the graph. The problem of computing the angular resolution of a graph is NP-hard (even the problem of deciding if $R = \pi/2$ for graphs with vertex degrees at most four is NP-hard).

The main focus in the early investigation [10, 15] was on bounding the angular resolution of a graph in terms of the maximum vertex degree d. The obvious upper bound is $R(d) \leq 2\pi/d$. A lower bound $R(d) \geq \Omega(1/d)$ has been proved [10] for many graphs including planar graphs and complete graphs. A lower bound $R(d) \geq \Omega(1/d^2)$ holds for all graphs [10]. If we insist on planar straight-line drawings, then $R(d) \geq \alpha^d$ for some constant $0 < \alpha < 1$ [15].

There was also a study of the optimization problem where the angular resolution of a given graph is maximized. Carlson and Eppstein [4] considered tree drawings such that all faces form (infinite) convex polygons. They showed that the optimal angular resolution can be computed in linear time and even the lengths of the tree edges may be chosen arbitrarily. In the recent paper, Eppstein and Wortman [9] considered graph drawing in the plane with faces drawn as centrally symmetric convex polygons. They found a polynomial time algorithm for computing a drawing maximizing the angular resolution.

Another direction in graph drawing with the angle resolution is *Lombardi drawing* where the graph edges are represented as circular arcs instead of straight-line segments [5, 6, 7]. Relaxing the condition of straight-line segments allows to achieve *perfect angular resolution* where the edges are equiangularly spaced around each vertex. The classes of graphs admitting Lombardi drawings are presented in [7] (and the algorithms for finding these drawings). Duncan *et al.* [6] found that unrooted trees can drawn with perfect angular resolution and polynomial area.

In this paper we also study the problem of maximizing the angular resolution using the force-directed approach. This idea is not new and some algorithms for optimizing angular resolution using the force-directed or spring-embedding approach [2, 5]. The common feature of these approaches is that the forces are directed from one vertex to another vertex. Our approach is different in the sense that we want to optimize the dislocation of a vertex. It can be viewed as a *perturbation problem* where a graph is embedded in the plane with straight-line edges and we want to find a new position of a given vertex by moving it at distance at

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most $r > 0$ (for a given r) and optimizing the incident angles. The perturbation problem is interesting in its own right. It is also interesting that the problem is related to the famous Fermat problem (which appears as a special case when the vertex degree is three).

Another reason why we introduce and study the perturbation problem is that the parameter r allows to control the strength of the angular resolution force applied to vertices. This works well if we use more than one force. For example, we applied it to the spring embedding with length constraints.

The paper is organized as follows. In Section 2 we briefly describe force-directed graph drawing and introduce Max-Min-Angle problem. In Section 3 we recall the classical Fermat problem that appears as the special case of the MAA problem. In Section 4 we provide a solution of the MMA problem for vertex degree two. Optimal solution for degree three vertices is provided in Section 5. The algorithm and its performance is discussed in Section 6. Finally we conclude in Section 7.

2   Spring Embedder and Problem Statement

Force-directed graph drawing is a popular technique and there is a growing literature on force-directed drawing algorithms, see the recent survey by Kobourov [14]. Eades [8] introduced a mechanical model for graph drawing. To achieve aesthetically pleasing layouts and capture the edge length constraints he applied attracting/repelling force between two vertices if the distance between them is less/greater than the desired length. He found that the Hookes Law (linear) springs are too strong when the vertices are far apart but the logarithmic force solves this problem. As initial embedding of the graph the algorithm places its vertices at random locations. The algorithm stops after a sufficient number of iterations. If a state of equilibrium is reached i.e. all forces are zero, then the graph embedding reaches the desired positioning in the plane and remains static. An example of such drawing is shown in Figure 1.
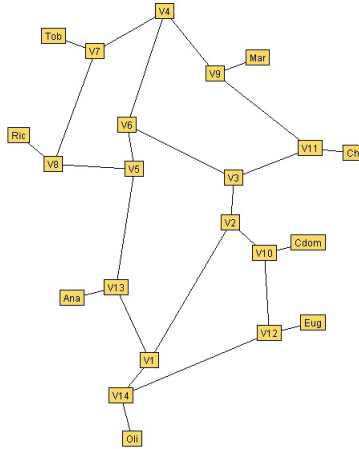

Figure 1: A drawing of Algae graph [16] produced by the spring embedder [1].

Fruchterman and Reingold [11] added a condition of "even vertex distribution" which is modeled by attractive forces between adjacent vertices and repulsive forces between all pairs of vertices. This increases the number of forces ($|V|^2$ repulsive forces for a graph $G = (V, E)$) and slows down the algorithm.

The algorithms by Eades [8] and Fruchterman and Reingold [11] are just two examples of force-directed graph drawing. There are many spring embedders nowadays [14]. We consider a new force-directed approach for angular resolution where the vertex displacement is optimized locally.

2.1   Problem Statement

We consider one step of the spring embedder where the vertices of a graph G are embedded in the plane and they are allowed to move slightly. The spring embedder may take into account several forces, for example the forces aiming to achieve the desired lengths of the edges of G. We introduce a force that aims to optimize,

for all vertices v, the angles between edges incident to v and embedded in the plane. Ideally, we want the edges to spread evenly around v. This may not be possible due to other constraints of the drawing (edge lengths, for example). Our goal is to maximize, for every v, the *smallest angle* between two edges incident to v. For this, we formulate our problem.

> **Max-Min-Angle (MMA) Problem**. Let G be a graph embedded in the plane. Let P be the position of a vertex v of G and let $A_1, A_2, \ldots, A_k$ be the positions of the adjacent vertices of P in G. Let $r > 0$ be a radius. Find a point P^* (the best position to move P within distance r) such that $|PP^*| \leq r$ and the smallest angle $\angle A_i P^* A_j$ is maximized.

Let Z be the circle centered at P and radius r. We will solve the MMA problem in Sections 4 and 5 by considering degree k of vertex v.

3   Fermat problem

When the degree of v is three, the MMA problem is related to the classical *Fermat problem*: Given triangle ABC in the plane, find a point F such that the total distance from the three vertices of the triangle to F is the minimum possible. The solution of the Fermat problem (called *Fermat point* or *Torricelli point*) depends on triangle ABC.
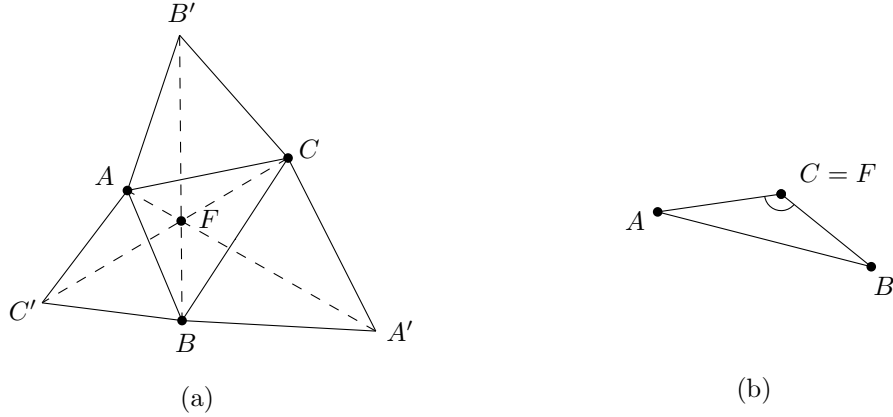

Figure 2: The Fermat point F in (a) Case 1 and (b) Case 2. (a) Three angles $\angle AFB, \angle BFC$, and $\angle CFA$ are equal. (b) The Fermat point is at C.

Case 1: All angles of triangle ABC are smaller than $120°$.
Construct three new regular triangles $ABC', AB'C$, and $A'BC$ out of the three sides of triangle ABC. Then the point F is the intersection of three lines AA', BB', and CC', see Figure 2(a). In this case the Fermat point F is coincident with the *first isogonic center* of the triangle [18], where the angles subtended by the sides of the triangle are all equal, i.e. $\angle AFB, \angle BFC, \angle CFA = 120°$.
Case 2: There exists an angle in triangle ABC greater than or equal to $120°$.
Only one angle of the triangle is greater than or equal to $120°$, say $\angle ACB \geq 120°$, see Figure 2(b) for example. Then the Fermat point is coincident with C.

4   The MMA problem

We solve the MMA problem depending on the degree of vertex v.

4.1   Vertex of Degree 2

Suppose that the degree of v is two. Let A and B be the positions of its adjacent vertices. The task is to find a point P^* with $|PP^*| \leq r$ maximizing angle $\theta = \min(\angle AP'B, \angle BP'A)$ (the angles are in the

counterclockwise order). Suppose that segment AB intersects circle Z. Obviously, $\theta = 180°$ is achieved by any point P^* in the intersection of AB and Z. The solution is unique if $AB \cap Z$ is a single point (for example, AB may be tangent to Z).

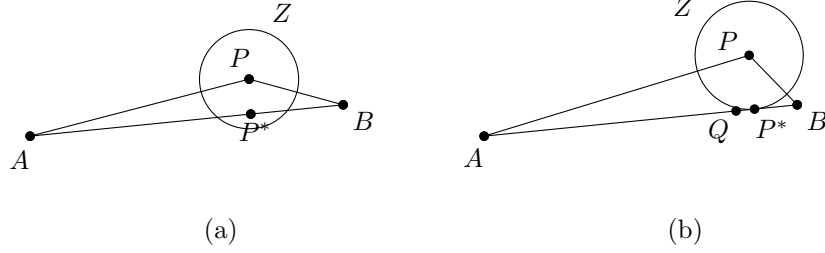(a)   (b)

Figure 3: The case where $AB \cap Z \neq \emptyset$. (a) $P^* = Q$ where Q is defined by Equation(1). (b) Point Q lies not in $AB \cap Z$.

Suppose that $AB \cap Z$ is a segment of positive length, see Figure 3(a). There are infinitely many positions of P^* (all in $AB \cap Z$ achieving $\theta = \pi$). If P is not on segment AB then any location of P^* will change distances to A and B. We would like to preserve the ratio $|AP|/|BP|$ and compute point Q in segment AB such that $|AQ|/|BQ| = |AP|/|BP|$. This implies

$$\frac{|AQ|}{|AB|} = \frac{|AP|}{|AP| + |BP|}. \tag{1}$$

Then the coordinates of point Q can be computed using

$$Q_x = A_x + \frac{|AP|}{|AP| + |BP|}(B_x - A_x), \tag{2}$$

$$Q_y = A_y + \frac{|AP|}{|AP| + |BP|}(B_y - A_y). \tag{3}$$

If point Q lies in circle Z then $P^* = Q$ as shown in Figure 3(a). Otherwise we select the endpoint of segment $AB \cap Z$ that is closer to Q, see Figure 3(b) for an example.

We assume now that AB and Z do not intersect.

Proposition 1 *If segment AB and circle Z do not intersect then P^* is on the boundary of Z. Furthermore the circle passing through A, B, and P^* is tangent to Z.*

Proof: Suppose $P^* = (x, y)$ where x and y are unknown. Consider circular arc AP^*B as shown in Figure 4(a). The angles $\angle AP'B$ are equal for all points P' from arc AP^*B (since the inscribed angle of a chord equals half of the central angle ,see Figure 8(b)). Therefore we can assume that P^* is on the boundary of Z. Furthermore arc AP^*B cannot intersect the boundary of Z twice since any point P' from the arc of Z cut off by arc AP^*B makes an angle larger than P^* makes, i.e. $\angle AP'B > \angle AP^*B$ as shown in Figure 4. Thus, arc AP^*B must be tangent to Z as shown in Figure 5(a). ∎

The MMA problem can be formulated now as the following problem dealing with only one angle.

Maximum Angle Problem. Let A, B, and P be points in the plane and a number $r > 0$ such that $|AP|, |BP| > r$. Compute a point P^* with $|PP^*| = r$ maximizing angle $\angle AP^*B$.

Let O be the center of the circle passing through A, B, and P^*. It suffices to find point O since point P^* is the intersection point of the circle Z and the line passing through points O and P. The first equation is $(P_x^* - P_x)^2 + (P_y^* - P_y)^2 = r^2$ and the second equation is linear

$$\begin{vmatrix} P_x^* & P_y^* & 1 \\ P_x & P_y & 1 \\ O_x & O_y & 1 \end{vmatrix} = 0,$$
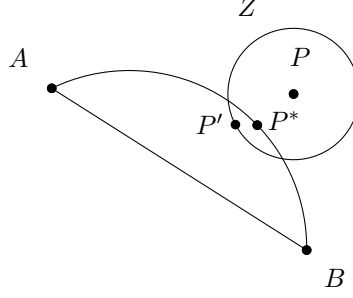
Figure 4: Arc AP^*B. Wrong location of P^* since $\angle AP'B > \angle AP^*B$.
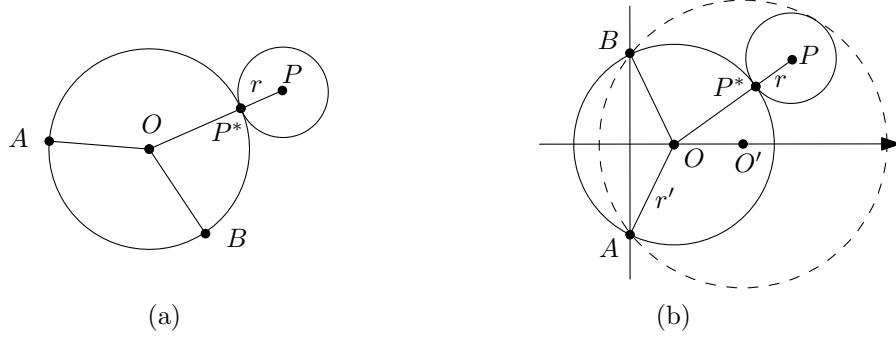

(a)   (b)

Figure 5: (a) The maximum angle problem. (b) The circles centered at O and O' are the two solutions of the system of equations (4) and (5).

where the subscripts denote the coordinates.

Using coordinate transformations we can assume without loss of generality that $A(0, -1)$ and $B(0, 1)$, see Figure 5(b). Then $|OA| = |OB|$ implies that $O(x, 0)$ where x is unknown. Let $r' = |OA|$ be the radius of the circle centered at O. Then $|OP| = r + r'$ and $|OA| = r'$. They can be written as

$$\begin{cases} (x - P_x)^2 + P_y^2 = (r + r')^2 & (4) \\ x^2 + 1 = r'^2 & (5) \end{cases}$$

Subtracting Equation (4) from Equation (5) , we obtain

$$2P_x x = P_x^2 + P_y^2 - 2r'r - r^2 - 1. \tag{6}$$

By plugging x from this equation into Equation of (5) we obtain a quadratic equation in the variable r'. There are two roots of the quadratic equation and they correspond to two circles shown in Figure 6. Therefore we take the smallest root of the quadratic equation.

Therefore we proved the folowing claim.

Proposition 2 *The maximum angle problem can be solved using a quadratic equation.*

5   Vertex of degree 3

In this Section we consider the case where the degree of v is three. Let A, B and C be the positions of its adjacent vertices in G. We consider the first case of the Fermat point of triangle ABC where all angles of ABC are less than $120°$.

Proposition 3 *If all angles of ABC are less than $120°$ and the Fermat point F of ABC lies within circle Z, then $P^* = F$.*
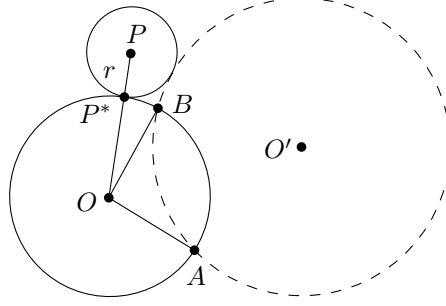
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Figure 6: An example of the maximum angle problem. The circles centered at O and O' are the two solutions of the system of equations (4) and (5).

Proof: Without loss of generality we assume that A, B, C are in counterclockwise order around P^*, see Figure 7 (a). Suppose to the contrary that P^* is a point inside Z different from F as shown in Figure 7 (b). The Fermat point lies in one of triangles AP^*B, BP^*C, or AP^*C. Suppose that it belongs to triangle AP^*C. Then $\angle AP^*C < \angle AFC = 120°$ and P^* is not the optimal point. Therefore $P^* = F$, see Figure 7 (c). ∎
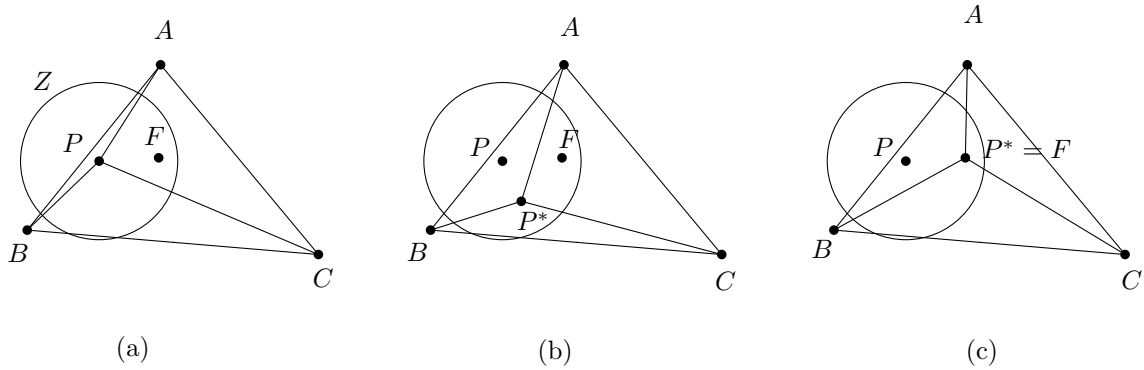

Figure 7: (a) Setting of Proposition 3. (b) Contradiction when $P^* \neq F$. (c) Points P^* and F coincide.

Note that in the case of Proposition 3 all angles around P^* are equal $120°$. In the rest of this section we consider cases where not all angles $\angle AP^*B, \angle BP^*C$, and $\angle AP^*C$ are equal. If the smallest angle of $\angle AP^*B, \angle BP^*C$, and $\angle AP^*C$ is unique, say it is $\angle AP^*B$, then it can be computed by solving the maximum angle problem for A, B using the method from the previous section. It remains to consider the case of two smallest angles.

5.1 Two smallest angles

In this section we assume that there are exactly two smallest angles from $\{\angle AP^*B, \angle AP^*C, \angle BP^*C\}$. First, we show that P^* must be on the boundary of circle Z.

Lemma 4 *If the degree of v is 3 and there are two smallest angles around P^* then P^* lies on the boundary of Z.*

Proof: We assume that
(i) A, B, C are in counterclockwise order around P^*, and
(ii) $\angle AP^*B = \angle BP^*C < \angle CP^*A$.
Suppose to the contrary that P^* lies inside Z as shown in Figure 8. Draw two circles, one passing through points A, B, P^* and the second passing through points B, C, P^*. Since these circles intersect by two points

B and P^*, then their interiors intersect by a lune. Since P^* is inside circle Z, then the intersection of the lune and the interior of Z is a non-empty set I, see the shaded area in Figure 8(a). For any point P' in I, $\angle AP'B > \angle AP^*B$ and $\angle BP'C > \angle BP^*C$ (this can be seen, for example, by the fact that the inscribed angle of a chord equals half of the central angle, see Figure 8(b)). We choose P' from I close enough to P^* such that $\angle AP'B, \angle BP'C < \angle CP'A$. Then the smallest angle around P' is greater than $\angle AP^*B$. Therefore P^* is not the solution of the MMA problem. Contradiction. ∎

(a)   (b)
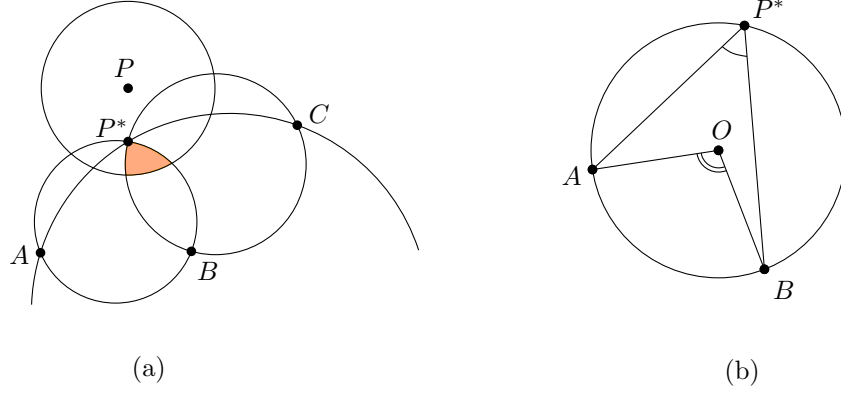
Figure 8: (a) For all points P' in the shaded area $\angle AP'B > \angle AP^*B$ and $\angle BP'C > \angle BP^*C$. (b) The inscribed angle $\angle AP^*B$ of chord AB equals half of the central angle $\angle AOB$.

The main result in this section is the following

Theorem 5 *If the degree of vertex v is three and there are two smallest angles around P^* then P^* can be computed by solving a polynomial equations of degree at most four.*
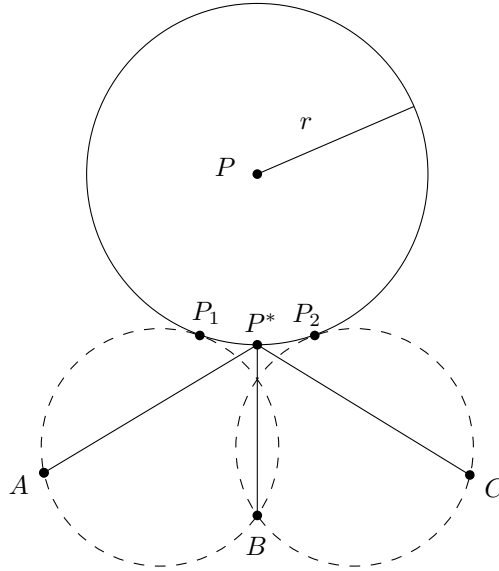

Figure 9: Points P_1 and P_2 are the points on circle Z maximizing angles $\angle AP_1B$ and $\angle BP_2C$, respectively. Point P^* is different from P_1 and P_2 and angles $\angle AP^*B = \angle BP^*C$.

Proof: First we show that it is possible that there are two smallest angles. An example where two smallest angles around P^* are equal is shown in Figure 9 where points P, P^* and B are collinear and points A and

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C are symmetric about the line PB. Therefore angles $\angle AP_2B < \angle AP^*B < \angle AP_1B$ and P^* is the solution of the MMA problem.

We now prove the theorem. Suppose that $\angle AP^*B = \angle BP^*C < \angle CP^*A$. We need to find coordinates of $P^*(x,y)$. By the law of cosines we write the equation $\cos(\angle AP^*B) = \cos(\angle BP^*C)$ as

$$\frac{|BP^*|^2 + |AP^*|^2 - |AB|^2}{2 \cdot |BP^*| \cdot |AP^*|} = \frac{|BP^*|^2 + |CP^*|^2 - |BC|^2}{2 \cdot |BP^*| \cdot |CP^*|} \tag{7}$$

$$|CP^*| \cdot (|BP^*|^2 + |AP^*|^2 - |AB|^2) = |AP^*| \cdot (|BP^*|^2 + |CP^*|^2 - |BC|^2) \tag{8}$$

Without loss of generality we can assume that $P = (0,0)$ and $A_y = C_y$. By Lemma 4 we assume $|PP^*| = r$. Then $x^2 + y^2 = r^2$ and

$$|CP^*|^2 = r^2 + C_x^2 + C_y^2 - 2C_x x - 2C_y y = a_1 x + a_2 y + a_3 \tag{9}$$

$$\frac{1}{2}(|BP^*|^2 + |AP^*|^2 - |AB|^2) = r^2 + A_x B_x + A_y B_y - (A_x + B_x)x - (A_y + B_y)y = a_4 x + a_5 y + a_6 \tag{10}$$

$$|AP^*|^2 = r^2 + A_x^2 + A_y^2 - 2A_x x - 2A_y y = b_1 x + b_2 y + b_3 \tag{11}$$

$$\frac{1}{2}(|BP^*|^2 + |P^*C|^2 - |BC|^2) = r^2 + B_x C_x + B_y C_y - (B_x + C_x)x - (B_y + C_y)y = b_4 x + b_5 y + b_6, \tag{12}$$

where $a_1 = -2C_x, a_2 = -2C_y, a_3 = r^2 + C_x^2 + C_y^2, a_4 = -(A_x + B_x), a_5 = -(A_y + B_y)$,
$a_6 = B_x A_x + B_y A_y + r^2$, $b_1 = -2Ax, b_2 = -2A_y, b_3 = r^2 + A_x^2 + Ay^2, b_4 = -(B_x + C_x), b_5 = -(By + Cy)$,
$b_6 = B_x C_x + B_y C_y + r^2$.

Take the square of both sides of Equation (8)

$$(a_1 x + a_2 y + a_3)(a_4 x + a_5 y + a_6)^2 = (b_1 x + b_2 y + b_3)(b_4 x + b_5 y + b_6)^2 \tag{13}$$

It can be written as

$$c_1 x^3 + c_2 y^3 + c_3 x^2 + c_4 y^2 + c_5 x^2 y + c_6 x y^2 + c_7 x + c_8 y + c_9 x y + c_{10} = 0 \tag{14}$$

where $c_1 = a_1 a_4^2 - b_1 b_4^2, c_2 = a_2 a_5^2 - b_2 b_5^2, c_3 = (2a_1 a_4 a_6 + a_3 a_4^2 - (2b_1 b_4 b_6 + b_3 b_4^2))$,
$c_4 = (2a_2 a_5 a_6 + a_3 a_5^2 - (2b_2 b_5 b_6 + b_3 b_5^2)), c_5 = (2a_1 a_4 a_5 + a_2 a_4^2 - (2b_1 b_4 b_5 + b_2 b_4^2))$,
$c_6 = (2a_2 a_4 a_5 + a_1 a_5^2 - (2b_2 b_4 b_5 + b_1 b_5^2)), c_7 = (2a_3 a_4 a_6 + a_1 a_6^2 - (2b_3 b_4 b_6 + b_1 b_6^2))$,
$c_8 = (2a_3 a_5 a_6 + a_2 a_6^2 - (2b_3 b_5 b_6 + b_2 b_6^2)), c_9 = 2(a_1 a_5 a_6 + a_2 a_4 a_6 + a_3 a_4 a_5 - (b_1 b_5 b_6 + b_2 b_4 b_6 + b_3 b_4 b_5))$,
$c_{10} = a_3 a_6^2$.

Using $y^2 = r^2 - x^2$ we reduce powers of y in Equation (14)

$$c_1 x^3 + y(c_2 r^2 + c_8) + c_3 x^2 + c_4 - c_4 x^2 + x^2 y(c_5 - c_2) + c_6 x - c_6 x^3 + c_7 x + c_8 y + c_9 x y + c_{10} = 0 \tag{15}$$

$$d_1 x^3 + d_2 x^2 + d_3 x + d_4 + (d_5 x^2 + d_6 x + d_7)y = 0 \tag{16}$$

where $d_1 = c_1 - c_6, d_2 = c_3 - c_4, d_3 = c_6 r^2 + c_7, d_4 = c_4 r^2 + c_{10}, d_5 = c_5 - c_2, d_6 = c_9, d_7 = c_2 r^2 + c_8$.

$$\left(d_1 x^3 + d_2 x^2 + d_3 x + d_4\right)^2 - ((d_5 x^2 + d_6 x + d_7)^2 \cdot (r^2 - x^2)) = 0 \tag{17}$$

Simplifying Equation (17) we get a sextic equation

$$e_1 x^6 + e_2 x^5 + e_3 x^4 + e_4 x^3 + e_5 x^2 + e_6 x + e_7 = 0 \tag{18}$$

where $e_1 = d_1^2 + d_5^2, e_2 = 2(d_1 d_2 + d_5 d_6), e_3 = (d_2^2 + 2d_1 d_3 - (d_5^2 r^2 - (d_6^2 + 2d_5 d_7)))$,
$e_4 = (2(d_1 d_4 + d_2 d_3) - (2(d_5 d_6 r^2 - d_6 d_7))), e_5 = ((d_3^2 + 2d_2 d_4) - ((d_6^2 + 2d_5 d_7)r^2 - d_7^2))$,
$e_6 = 2d_3 d_4 - (2d_6 d_7 r^2), e_7 = d_4^2 - d_7^2 r^2$.

By Lemma 6 the sextic equation has a quadratic factor. Dividing by it, we obtain a polynomial of degree four (Equation (19),(20), or (21)) and the theorem follows. ∎

Lemma 6 *The polynomial equation (18) has factor $x^2 + A_y^2 - r^2$.*

Proof: We consider 3 cases.

Case I: $A_y = r$. Then $e_5 = e_6 = e_7 = 0$ and the polynomial (18) has factor x^2. The polynomial is reduced to the polynomial of degree 4

$$e_1 x^4 + e_2 x^3 + e_3 x^2 + e_4 x = 0. \tag{19}$$

Case II: $A_y^2 - r^2 > 0$. We scale the coordinates such that $A_y^2 - r^2 = 1$. Then $e_6 = e_4 - e_2$, $e_7 = e_3 - e_1$ and the polynomial (18) has factor $x^2 + 1$. The polynomial is reduced to the polynomial of degree 4

$$e_1 x^4 + e_2 x^3 + (e_3 - e_1) x^2 + (e_4 - e_2) x + e_5 - (e_3 - e_1) = 0. \tag{20}$$

Case III: $A_y^2 - r^2 < 0$. We scale the coordinates such that $A_y^2 - r^2 = -1$. Then $e_6 = e_4 + e_2$, $e_7 = e_3 + e_1$ and the polynomial (18) has factor $x^2 - 1$. The polynomial is reduced to the polynomial of degree 4

$$e_1 x^4 + e_2 x^3 + (e_3 + e_1) x^2 + (e_4 + e_2) x + e_5 + (e_3 + e_1) = 0. \tag{21}$$

The lemma follows. ∎

6 Algorithm

In this Section we discuss how to modify the spring embedder in order to take into account the angles between embedded edges. Let $G = (V, E)$ be an input graph. At the initial step the spring embedder randomly places the vertices of G in the plane. Then, it iterates a simultaneous motion of the vertex positions based on one or several forces (springs). We describe a new force using angle optimization. For every vertex v with a position P in the plane, the algorithm computes a new position P^* and uses vector PP^* as a force applied to vertex v.

Angle Optimization Algorithm

Input: Graph $G = (V, E)$ embedded in the plane and radius r.

Output: New embedding of graph G where each vertex is translated within distance r.

For each vertex $v \in V$ do the following steps.

1. Let P be the position of v in the plane. We compute P^* as follows. Every time P^* is assigned, we proceed to the next vertex v.

2. If the degree of v is at most one then set $P^* = P$.

3. Suppose that the degree of v is equal to two. Let A and B be the positions of vertices adjacent to v.

   (a) Suppose $AB \cap Z \neq \emptyset$. Compute point Q using Equations (2) and (3). If $|PQ| \leq r$ then $P^* = Z$; otherwise assign P^* to the endpoint of segment $AB \cap Q$ that is closer to Q.

   (b) If $AB \cap Z = \emptyset$ then set compute P^* using Proposition 2.

4. Suppose that the degree of v is equal to three. Let A, B, and C be the positions of vertices adjacent to v. Compute Fermat point F of triangle ABC. If $|PF| \leq r$ then set $P^* = F$. Otherwise, for each segment $ab \in \{AB, AC, BC\}$, compute point P_{ab} maximizing angle $\angle a P_{ab} b$ (Section 4.1). If angle $\angle a P_{ab} b$ is the smallest angle from $\{\angle A P_{AB} B, \angle A P_{AC} C, \angle B P_{BC} C\}$ then set $P^* = P_{ab}$. Otherwise compute P^* as the best solution using two smallest angles for every two segments from $\{AB, AC, BC\}$ (Section 5.1).

5. For the remaining vertices of degree at least four, apply the following grid approach. Pick a grid stepsize δ, for example $\delta = r/3$. Consider a grid with the origin at P. For every grid point Q with $|PQ| \leq r$, compute the smallest angle α_Q if P is moved to Q. Find Q maximizing α_Q. Assign $P^* = Q$.
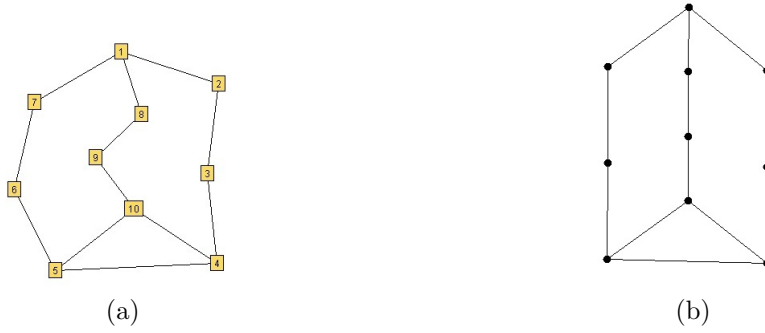
Figure 10: Graph T_{10} from [17]. (a) Drawing by Spring Embedder [1]. (b) Drawing by our program with angle optimization.

We implemented this algorithm[1] and run it on several graphs. First, we tested the algorithm on graph T_{10} from [17] since it contains vertices of degree two. The program draws T_{10} with angles optimized, see Figure 10 (b). It can be compared with the drawing of the original embedder [1], see Figure 10 (a).

We also tested the algorithm on the phylogenetic networks for Algae example [16]. The drawing of the Algae network by the spring embedder [1] is shown in Figure 1. It can be compared with our drawings in Figure 11. In all drawings (in Figures 1 and Figure 11) the edge length constraints are satisfied but the angle resolution in drawings in Figure 11 is significantly larger.

shown in Figure 11 (a) uses the weighted version of the graph and the drawing in Figure 11 (b) uses the graph with intermediate points on the edges.
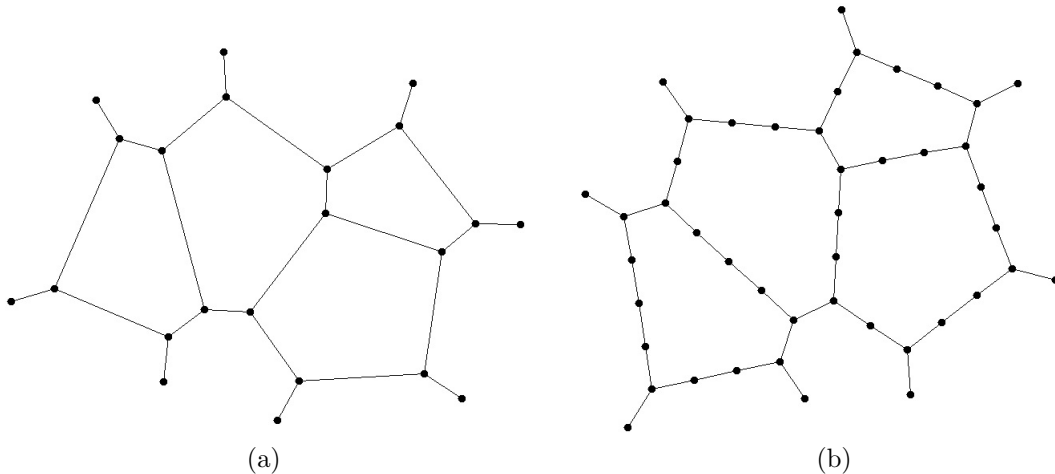

Figure 11: Phylogenetic network for the Algae example [16]. (a) Weighted graph. (b) Graph with intermediate points on the edges.

Finally, we run our program on the well-known graphs: the Petersen graph [13], the Heawood graph [12], and the Herschel graph [3], see Figure 12. The Petersen graph is drawn with two crossings (the Petersen graph is in fact the smallest 2-crossing cubic graph), see Figure 12(a). The Heawood graph has crossing number three (it is actually the smallest 3-crossing cubic graph) and our program found a drawing with exactly three crossings, see Figure 12(b).

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[1]Demo is available at `http://www.utdallas.edu/~sxb027100/soft/AngleOpt/`.
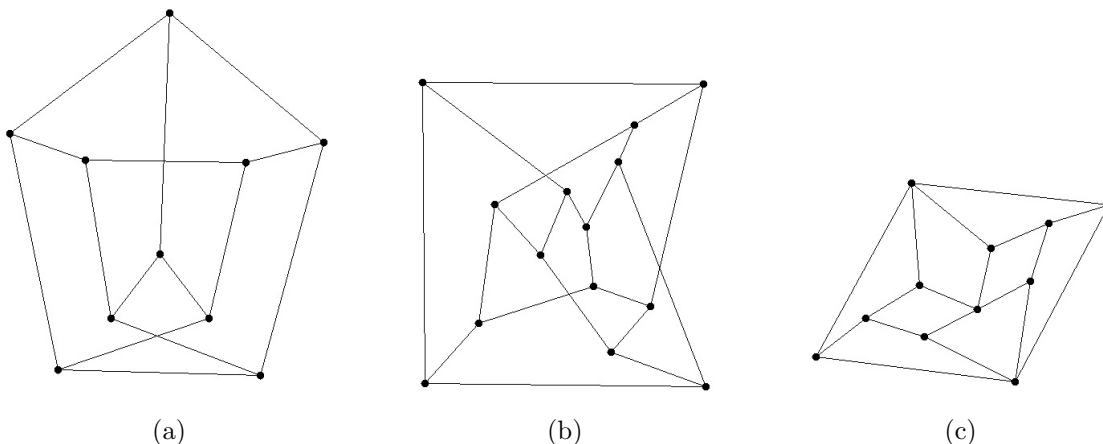
Figure 12: (a) The Petersen graph. (b) The Heawood graph. (c) The Herschel graph.

7 Conclusion

We proposed a novel approach to the problem of optimizing the angular resolution of a drawing. It has been applied to the spring embedder and the results with good angular resolution were obtained. It is known that the running time of the spring embedder can increase with the size of the graph. Therefore it is important to perform better the intermediate steps. The optimal solution of the MMA problem provides an opportunity to decrease the number of iterations of the spring embedder.

The main result of this paper states that a vertex of degree at most three can be displaced optimally by solving a polynomial equation of degree at most four (which is interesting since the straightforward approach leads to a polynomial of degree six). A special case of MMA problem for degree three verices is the classical Fermat problem and the Fermat point is the solution for the special case. An interesting question for future research is to find the algebraic complexity of MMA problem for higher vertex degrees.

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